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ADDENDUM

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Abstract. The class of external fields for which the causal Green functions of the Klein–Gordon and Dirac equations can be calculated exactly by means of the methods of integrals of motion and coherent states is shown. Several important examples are considered.

In recent papers by Dodonov *et al* (1975a, b, c, to be referred to as I, II, III) the general method of calculating the Green functions of the Schrödinger equation by means of the integrals of the motion and coherent states was suggested. This method gives the equations for the Green function $G(z_1; z_2; t)$ of any dynamical system with N degrees of freedom. (As usual, we mean by the dynamical system any system described by an equation of the type $\mathrm{i}\,\partial\psi/\partial t = \hat{H}\psi$, $\hat{H}$ being an arbitrary operator, t being an arbitrary parameter.)

In an arbitrary z-representation these equations are (see equations (1.12)–(1.13) of II, III or equations (1*a*)–(1*b*) of I in the case of the coordinate representation):

$$\begin{aligned} &\hat{F}_{j(1)}G(z_1; z_2; t) = \hat{f}_{j(2)}G(z_1; z_2; t) \\ &j = 1, 2, \ldots 2N; \qquad z = (z^{(1)}, z^{(2)}, \ldots z^{(N)}) \end{aligned} \tag{1}$$

here $\hat{f}_j, j = 1, 2, \ldots 2N$, are $2N$ quite arbitrary independent operators, and $\hat{F}_j$ are the quantum integrals of the motion coinciding with $\hat{f}_j$ at the initial moment $t = 0$. The symbol $\hat{I}_{(k)}G(z_1; z_2; t)(k = 1, 2)$ means that the operator $\hat{I}$ acts on the function G as the function of only the first ($k = 1$) group of arguments (z_1) or the second ($k = 2$) one (z_2), while other variables should be considered as parameters; $\hat{f}^T$ means the transposed operator. The equations (1) were used in I–III to obtain the explicit expressions for the Green functions of the most general non-stationary multi-dimensional non-relativistic quadratic system in various representations and for the Green function of the non-stationary oscillator with the additional term gx^{-2} (the so called singular oscillator); this last problem was also considered using another method by Dodonov *et al* (1974a, b). The equations which are the special case of equations (1) were also applied by Landovitz (1975) for calculating the Green function (in the coordinate representation) of a very special example of general quadratic systems: an oscillator in uniform constant magnetic and periodic electric fields. This problem was earlier solved by Malkin and Man'ko (1970). In more recent interesting papers by Campbell *et al* (1976a, b) some equations similar to equations (1) were used for solving the eigenvalue problem for various operators. However, these authors did not use the most general equations, and did not clarify the meaning of the operators entering into equations of the type (1) as the integrals of the motion. Actually the equations by Campbell *et al* are the consequences of equations (1), since every solution of the Schrödinger equation $\psi(z; z_0; t)$ depending on N parameters $z_0^{(1)}, z_0^{(2)}, \ldots z_0^{(N)}$ can be considered as the Green function in an appropriate z, z_0-representation, and vice versa.

In all the papers mentioned above only non-relativistic problems were considered. In the present article we want to apply the method of integrals of the motion to the relativistic Klein–Gordon and Dirac equations and to find the Green functions of these equations for some new types of external fields.

We shall consider mainly the Klein–Gordon equation. The specific features of Dirac's equation will be discussed at the end. The Green function of the Klein–Gordon equation satisfies the equation

$$\begin{aligned}&[(\hat{p}^\mu - eA^\mu)(\hat{p}_\mu - eA_\mu) - m^2]G(q''; q') = \delta(q''-q')\\&q = (q^0, q^1, q^2, q^3); \qquad \hat{p}_\mu = \mathrm{i}\,\partial/\partial q^\mu; \qquad \hbar = c = 1\\&g^{00} = -g^{11} = -g^{22} = -g^{33} = 1; \qquad g^{\mu\nu} = 0, \mu \neq \nu.\end{aligned} \tag{2}$$

A^μ is the four-dimensional vector potential of the external field.

As was shown by Fock (1937) and Schwinger (1951), $G(q''; q')$ can be expressed in terms of the integral over the proper time:

$$G(q''; q') = \frac{1}{2\mathrm{i}} \int_0^\infty \exp[-\tfrac{1}{2}\mathrm{i}(m^2 - \mathrm{i}\epsilon)s]g(q''; q'; s)\,\mathrm{d}s \qquad \epsilon \to +0 \tag{3}$$

where the function $g(q''; q'; s)$ is the Green function (the propagator) of the Schrödinger equation

$$\begin{aligned}&\left(\mathrm{i}\frac{\partial}{\partial s} - \hat{H}\right)g(q''; q'; s) = \mathrm{i}\,\delta(s)\,\delta(q''-q');\\&\hat{H} = -\tfrac{1}{2}(\hat{p}^\mu - eA^\mu)(\hat{p}_\mu - eA_\mu).\end{aligned} \tag{4}$$

Equation (4) can be solved exactly using the method of I–III for the two main types of potential A^μ.

(*a*) In the first case the contravariant components of the vector potential are as follows:

$$A^0 = \tfrac{1}{2}\tilde{f}(t-z); \qquad A^3 = -\tfrac{1}{2}\tilde{f}(t-z); \qquad \tilde{f} = \text{constant}; \qquad q = (t, x, y, z)$$

$$A^1 = -\tfrac{1}{2}\omega_1(\xi_4)y + \omega_3(\xi_4)x + F_1(\xi_4) + \sum_k \lambda_k^{(1)}[\hat{a}_k^{(1)}\,\mathrm{e}^{-\mathrm{i}\Omega_k\xi_4} + (\hat{a}_k^{(1)})^\dagger\,\mathrm{e}^{\mathrm{i}\Omega_k\xi_4}]; \tag{5}$$

$$A^2 = \tfrac{1}{2}\omega_2(\xi_4)x + \omega_4(\xi_4)y + F_2(\xi_4) + \sum_k \lambda_k^{(2)}[\hat{a}_k^{(2)}\,\mathrm{e}^{-\mathrm{i}\Omega_k\xi_4} + (\hat{a}_k^{(2)})^\dagger\,\mathrm{e}^{\mathrm{i}\Omega_k\xi_4}]; \qquad \xi_4 = t - z.$$

This potential describes in the general case a uniform stationary electric field $\tilde{f}$ directed parallel to the z axis, a non-uniform non-stationary magnetic field $\tfrac{1}{2}(\omega_1 + \omega_2)$ directed along the same axis, non-uniform crossed electric and magnetic fields in the xy-plane, an arbitrary classical plane wave moving along the z axis, and the quantized field of photons with the frequencies Ω_k and two different polarizations moving in the same z direction. The model in which the interaction of a charged particle with the quantized radiation field is described by introducing the operator terms ($\hat{a}_k^{(j)}$ and $(\hat{a}_k^{(j)})^\dagger$ are the annihilation and creation operators) into the vector potential of the external field was suggested by Dirac (1946) (see also Sen Gupta 1952). The functions ω_j, $j=1,2,3,4$ and F_k, $k=1,2$, may be arbitrary functions of ξ_4. However, the Hamiltonian (4) with the potential (5) can be considered as a quadratic Hamiltonian.

Indeed, introducing the new variables

$$\xi_3 = p^0 - p^3; \qquad \eta_3 = -\tfrac{1}{2}(t+z); \qquad \eta_4 = \tfrac{1}{2}(p^0+p^3)$$
$$[\hat{\eta}_j, \hat{\xi}_k] = \mathrm{i}\delta_{jk}; \qquad j,k=3,4$$

(these variables were suggested by Dirac 1949) one can rewrite the Hamiltonian of equation (4) with the potential (5) as follows:

$$\hat{H} = \hat{H}_\parallel + \hat{H}_\perp; \qquad \hat{H}_\parallel = -\hat{\xi}_3\hat{\eta}_4 + \tfrac{1}{2}f(\hat{\xi}_4\hat{\eta}_4 + \hat{\eta}_4\hat{\xi}_4); \qquad f = e\tilde{f}$$
$$\hat{H}_\perp = \tfrac{1}{2}(\hat{p}^1 - eA^1)^2 + \tfrac{1}{2}(\hat{p}^2 - eA^2)^2.$$

Following the general method of I we are to find integrals of the motion of equation (4). One can check that the operators

$$\hat{I}_3 = \hat{\xi}_3; \qquad \hat{I}_4 = \frac{1}{f}(1 - e^{fs})\hat{\xi}_3 + e^{fs}\hat{\xi}_4$$

are integrals of the motion for equation (4). Therefore the function $g(s)$ can be represented in the form

$$g(\{q''_\perp\}, \xi_3, \xi_4; \{q'_\perp\}, I_3, I_4; s)$$
$$= e^{-fs/2}\delta(\xi_3 - I_3)\delta(\xi_4 - f^{-1}(1-e^{-fs})I_3 - e^{-fs}I_4)\, g_\perp(\{q''_\perp\}; \{q'_\perp\}; s) \tag{6}$$

where the symbol $\{q_\perp\}$ designates the transverse coordinates of the particle x, y and the set of quantum numbers describing photons; the function $g_\perp(s)$ is the Green function of the two-dimensional Schrödinger equation with the transverse Hamiltonian $\hat{\tilde{H}}_\perp$ which can be obtained from $\hat{H}_\perp$ by means of replacing the operator $\hat{\xi}_4$ by the c-number function $\chi(s; I_3, I_4) = f^{-1}(1-e^{-fs})I_3 + e^{-fs}I_4$. Thus the problem is reduced to solving the Schrödinger equation with the non-stationary quadratic Hamiltonian.

The integral (3) in the case under study can be easily calculated due to the δ function in equation (6). Therefore the Green function of the Klein–Gordon equation in the $\{q_\perp\}$, ξ_3, ξ_4-representation is

$$G(\{q''_\perp\}, \xi_3, \xi_4; \{q'_\perp\}, I_3, I_4)$$
$$= \frac{u^{-1/2}\theta(u^{\operatorname{sgn} f})}{2\mathrm{i}|I_3 - fI_4|}\delta(\xi_3 - I_3)\exp\left(\frac{\mathrm{i}m^2}{2f}\ln u\right) g_\perp(\{q''_\perp\}; \{q'_\perp\}; -f^{-1}\ln u);$$
$$u = (I_3 - f\xi_4)/(I_3 - fI_4); \tag{7}$$
$$\theta(u) = \int_0^1 \delta(x-u)\,\mathrm{d}x = \begin{cases} 1, & 0<u<1 \\ \tfrac{1}{2}, & u=0,1 \\ 0, & 1/u<1. \end{cases}$$

If $f=0$, then equation (7) becomes:

$$G=\frac{\delta(\xi_3-I_3)}{2\mathrm{i}|I_3|}\Theta\left(\frac{\xi_4-I_4}{I_3}\right)\exp\left(-\frac{\mathrm{i}m^2}{2I_3}(\xi_4-I_4)\right)g_\perp\left(\{q''_\perp\};\{q'_\perp\};\frac{\xi_4-I_4}{I_3}\right);\tag{7a}$$

$$\Theta(x)=\begin{cases}1, & x>0\\ \frac{1}{2}, & x=0\\ 0, & x<0.\end{cases}$$

The formula (7) was not known up to now, although various special cases of the potential (5) were considered by many authors (for references see e.g. Nickle and Beers 1972 and Dodonov *et al* 1976). To obtain the explicit form of the function $g_\perp(s)$ one should only substitute the concrete values of the parameters determining the quadratic Hamiltonian $\hat{H}_\perp$ into the general formulae for the Green functions of quadratic Hamiltonians given in I–III. (Needless to say exact results can be obtained not only for the potential (5), but for all potentials which can be obtained from (5) by means of Lorentz or gauge transformations.)

The function $g_\perp(s)$ in the absence of the quantized field ($\lambda_k=0$) was given by Dodonov *et al* (1975d, 1976). Therefore here we consider briefly the case $\lambda_k\neq 0$. Let us suppose for simplicity that $f=\omega_3=\omega_4=0$; $\omega_1=\omega_2=\omega=$ constant, and that all coupling constants λ_k are small. Then one can take into account only linear (with respect to λ_k) terms (provided $\omega\neq I_3\Omega_k$, see below).

In this approximation the Green function is factorized, so that it is sufficient to consider only two modes with different polarizations and the same frequency Ω. To eliminate the exponential terms $\exp(\pm\mathrm{i}\Omega\xi_4)$ we make the following canonical transformation $\psi_{\mathrm{new}}=\exp[\mathrm{i}\xi_4\Omega(\hat{a}_1^\dagger\hat{a}_1+\hat{a}_2^\dagger\hat{a}_2)]\psi_{\mathrm{old}}$. It is very convenient to introduce the operator

$$\hat{a}_0=(2\omega)^{-1/2}[(\hat{p}_x+\tfrac{1}{2}\omega\hat{y})+\mathrm{i}(\hat{p}_y-\tfrac{1}{2}\omega\hat{x})].\tag{8}$$

The eigenfunctions of this operator are coherent states of a charged particle in a uniform magnetic field. The properties of these sates were discussed in detail by Malkin and Man'ko (1968). Having made all substitutions and transformations one can obtain from equation (2) the following equation (we suppose that $\lambda^{(1)}=\lambda^{(2)}$):

$$\begin{aligned}\Big(2\mathrm{i}I\frac{\partial}{\partial\xi}-\{&\omega(\hat{a}_0\hat{a}_0^\dagger+\hat{a}_0^\dagger\hat{a}_0)+I\Omega(\hat{a}_1^\dagger\hat{a}_1+\hat{a}_2^\dagger\hat{a}_2)+m^2-\lambda(2\omega)^{1/2}[(\hat{a}_0+\hat{a}_0^\dagger)(\hat{a}_1+\hat{a}_1^\dagger)\\ &+\mathrm{i}(\hat{a}_0-\hat{a}_0^\dagger)(\hat{a}_2+\hat{a}_2^\dagger)]-(2\omega)^{1/2}[F_1(\xi)(\hat{a}_0+\hat{a}_0^\dagger)+\mathrm{i}F_2(\xi)(\hat{a}_0-\hat{a}_0^\dagger)]\\ &+2\lambda[F_1(\xi)(\hat{a}_1+\hat{a}_1^\dagger)+F_2(\xi)(\hat{a}_2+\hat{a}_2^\dagger)]+F_1^2(\xi)+F_2^2(\xi)\}\Big)\\ &\times G(q'';q')=\delta(q''-q');\qquad \xi\equiv\xi_4.\end{aligned}\tag{9}$$

(We have replaced the integral of the motion $\hat{I}_3$ by its eigenvalue I, and the operator $\hat{\eta}_4$ by the operator $\partial/\partial\xi$.) Using the general formula for quadratic Hamiltonians given in the papers by Dodonov *et al* (1975b, c), one can obtain the following expression in the coherent states representation:

$$
\begin{aligned}
&G(\alpha_0,\alpha_1,\alpha_2,\xi,I;\beta_0,\beta_1,\beta_2,\xi',I')\\
&\quad=\frac{1}{2\mathrm{i}}\delta(I-I')\exp\Big[-\tfrac{1}{2}\mathrm{i}s(m^2+\omega)+\alpha_0\beta_0\,\mathrm{e}^{-\mathrm{i}\omega s}+\mathrm{e}^{-\mathrm{i}\tilde{\Omega}s}(\alpha_1\beta_1+\alpha_2\beta_2)\\
&\qquad+\frac{\lambda(\omega/2)^{1/2}}{\omega+\tilde{\Omega}}(1-\mathrm{e}^{-\mathrm{i}(\omega+\tilde{\Omega})s})[\beta_0(\beta_1+\mathrm{i}\beta_2)+\alpha_0(\alpha_1-\mathrm{i}\alpha_2)]\\
&\qquad+\frac{\lambda(\omega/2)^{1/2}}{\omega-\tilde{\Omega}}(\mathrm{e}^{-\mathrm{i}\tilde{\Omega}s}-\mathrm{e}^{-\mathrm{i}\omega s})[\alpha_0(\beta_1-\mathrm{i}\beta_2)+\beta_0(\alpha_1+\mathrm{i}\alpha_2)]\\
&\qquad+\mathrm{i}\lambda\,\mathrm{e}^{-\mathrm{i}\tilde{\Omega}s}(\alpha_1W_1+\mathrm{i}\alpha_2W_2)+\mathrm{i}\lambda(\beta_1W_1^*-\mathrm{i}\beta_2W_2^*)\\
&\qquad-\mathrm{i}\lambda\frac{\omega}{2}\left(\frac{W_0\,\mathrm{e}^{-\mathrm{i}\omega s}}{\omega-\tilde{\Omega}}(\alpha_1+\mathrm{i}\alpha_2)+\frac{W_0^*\,\mathrm{e}^{-\mathrm{i}\tilde{\Omega}s}}{\omega+\tilde{\Omega}}(\alpha_1-\mathrm{i}\alpha_2)\right)\\
&\qquad+\mathrm{i}(\omega/2)^{1/2}(\alpha_0W_0\,\mathrm{e}^{-\mathrm{i}\omega s}+\beta_0W_0^*)\\
&\qquad-\mathrm{i}\lambda\frac{\omega}{2}\left(\frac{W_0^*}{\omega-\tilde{\Omega}}(\beta_1-\mathrm{i}\beta_2)+\frac{W_0\,\mathrm{e}^{-\mathrm{i}(\omega+\tilde{\Omega})s}}{\omega+\tilde{\Omega}}(\beta_1+\mathrm{i}\beta_2)\right)\\
&\qquad-\frac{1}{2}\int_0^s\dot{W}_0^*(\omega W_0+\mathrm{i}\dot{W}_0)\,\mathrm{d}\tau-\frac{1}{2}\sum_{j=0}^{2}(|\alpha_j|^2+|\beta_j|^2)\Big];\\
&W_0(s)=\int_0^s\mathrm{e}^{\mathrm{i}\omega\tau}[F_1(\xi'+I\tau)-\mathrm{i}F_2(\xi'+I\tau)]\,\mathrm{d}\tau;\\
&W_1(s)=\int_0^s\mathrm{e}^{\mathrm{i}\tilde{\Omega}\tau}[\tilde{\Omega}F_1(\xi'+I\tau)-\mathrm{i}\omega F_2(\xi'+I\tau)]\,\mathrm{d}\tau;\\
&W_2(s)=\int_0^s\mathrm{e}^{\mathrm{i}\tilde{\Omega}\tau}[\omega F_1(\xi'+I\tau)-\mathrm{i}\tilde{\Omega}F_2(\xi'+I\tau)]\,\mathrm{d}\tau;\\
&s=(\xi-\xi')/I;\qquad \tilde{\Omega}=I\Omega;\qquad I=p^0-p^3.
\end{aligned}
\tag{10}
$$

An asterisk indicates complex conjugate. $F_j(\xi), j=1,2$ are the classical parts of the plane wave. α_n and β_n, $n=0,1,2$ are eigenvalues of the operators $\hat{a}_n$. The function (10) can be also considered as the transition amplitude from the initial coherent state $|\beta_0,\beta_1,\beta_2\rangle$ into the final state $|\alpha_0^*,\alpha_1^*,\alpha_2^*\rangle$.

Equation (10) is not valid in the resonance case, when $\omega=\pm I\Omega$. This case will be considered elsewhere.

(*b*) The second case corresponds to external fields which can reduce the Klein–Gordon equation to the problem of the singular oscillator. There are several different types of potentials of this kind. For example:

$$
\begin{aligned}
&eA^1=-y(\tfrac{1}{2}\omega(\xi)+\gamma\rho^{-2});\qquad eA^2=x(\tfrac{1}{2}\omega(\xi)+\gamma\rho^{-2});\\
&eA^0=eA^3=\tfrac{1}{2}\omega_0(\xi)\rho^2+\gamma_0\rho^{-2};\qquad \rho^2=x^2+y^2;\qquad \xi=t-z
\end{aligned}
\tag{11a}
$$

or

$$
A^1=A^2=0;\qquad eA^0=eA^3=\tfrac{1}{2}\omega(\xi)x^2+\gamma x^{-2};
\tag{11b}
$$

ω and ω_0 may be arbitrary functions of ξ; γ, γ_0 are constants. One can easily check that the function A^0 is contained in equation (1) only in the combination $\hat{I}A^0(\hat{I}=\hat{p}^0-\hat{p}^3)$, the operator $\hat{I}$ being the integral of the motion. We write the explicit expression for the Green function in the special case $A^0 = 0$; ω = constant (otherwise the formulae are too cumbersome). Introducing cylindrical coordinates and using the previous results of Dodonov *et al* (1974a) relating to the Schrödinger equation, one obtains:

$$G(\rho_2, \phi_2, I_2, \xi_2; \rho_1, \phi_1, I_1, \xi_1)$$
$$= \frac{\omega\delta(I_2 - I_1)}{8\pi \mathrm{i} \sin(\frac{1}{2}\omega s)} \exp[-\tfrac{1}{2}\mathrm{i}m^2 s + \tfrac{1}{4}\mathrm{i}\omega \cot(\tfrac{1}{2}\omega s)(\rho_2^2 + \rho_1^2)]$$
$$\times \sum_{l=-\infty}^{\infty} \exp[\mathrm{i}l(\phi_2 - \phi_1) + \tfrac{1}{2}\mathrm{i}\omega s(l - \gamma) - \tfrac{1}{2}\mathrm{i}\pi(1 + |l - \gamma|)]\, \mathrm{J}_{|l-\gamma|}\left(\frac{\omega\rho_1\rho_2}{2 \sin(\frac{1}{2}\omega s)}\right);$$
$$\tan\phi = y/x; \qquad s = (\xi_2 - \xi_1/I_2). \tag{12}$$

This function describes the motion of a charged scalar particle in the field which is the combination of the uniform magnetic field ω and the field of the infinitely long and infinitely thin solenoid parallel to the uniform magnetic field. The magnetic flux created by the solenoid is equal to $2\pi\gamma$. If $\gamma' = \gamma + n$, n being an integer, the functions $G(\gamma')$ and $G(\gamma)$ differ only by the phase factor $\exp[\mathrm{i}n(\phi_2 - \phi_1)]$. Exact solutions and Green functions can be also obtained for the potentials with the singular terms of the type $\gamma(\phi)\rho^{-2}$ (for the Schrödinger equation this was shown by Dodonov *et al* 1974b).

In conclusion we wish to make some remarks concerning Dirac's equation. It can be transformed by means of the well known procedure into the second-order equation which is similar to equation (1) with the only difference being that there is the additional term $-\frac{1}{4}\mathrm{i}eF_{\mu\nu}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$ in the left-hand side ($F_{\mu\nu}$ is the electromagnetic field tensor; γ^μ, $\mu = 0, 1, 2, 3$ are Dirac matrices). Therefore the Green function of Dirac's equation can be obtained easily from that of the corresponding Klein–Gordon equation, provided $F_{\mu\nu}$ depends only on the proper time. This restriction is very significant: in case (a) we have to deal only with constant uniform fields (ω_j = constant), since otherwise $F_{\mu\nu}$ would contain terms of the type $y\ \partial\omega(\xi)/\partial t$. Case ($b$) must be excluded for the same reason.

References

Nickle H H and Beers B L 1972 *J. Phys. A: Gen. Phys.* **5** 1658–63
Campbell W B, Finkler P, Jones C E and Misheloff M N 1976a *Nuovo Cim.* B **31** 219–24
—— 1976b *Ann. Phys., NY* **96** 286–302
Dirac P A M 1946 *Commun. Dublin Inst. Adv. Stud. Ser.* A **3** 14–9
—— 1949 *Rev. Mod. Phys.* **21** 392–9
Dodonov V V, Malkin I A and Man'ko V I 1974a *Physica* **72** 597–615
—— 1974b *Nuovo Cim.* B **24** 46–56
—— 1975a *J. Phys. A: Math. Gen.* **8** L19–22
—— 1975b *Teor. Mat. Fiz.* **24** 164–76
—— 1975c *Int. J. Theor. Phys.* **14** 37–54
—— 1975d *Lett. Nuovo Cim.* **14** 241–4
—— 1976 *Physica* A **82** 113–33
Fock V A 1937 *Sow. Phys.* **12** 404–25
Landovitz L F 1975 *Phys. Rev.* A **11** 67–70
Malkin I A and Man'ko V I 1968 *Zh. Eksp. Teor. Fiz.* **55** 1014–25
—— 1970 *Zh. Eksp. Teor. Fiz.* **59** 1746–58
Schwinger J 1951 *Phys. Rev.* **82** 664–79
Sen Gupta N D 1952 *Bull. Math. Soc. (Calcutta)* **44** 132–6